

# Catalytic Conversion Probabilities for Bipartite Pure States

S. Turgut\*

Department of Physics, Middle East Technical University, 06531, ANKARA, TURKEY

For two given bipartite-entangled pure states, an expression is obtained for the least upper bound of conversion probabilities using catalysis. The attainability of the upper bound can also be decided if that bound is less than one.

PACS numbers: 03.67.Mn, 03.65.Ud

## INTRODUCTION

A major problem in quantum information theory is to understand the conditions for transforming a given entangled state into another desired state by using only local quantum operations assisted with classical communication (LOCC). Significant development has been achieved for the case of pure bipartite states. Bennett *et al.* have shown that for the asymptotic case, where essentially an infinite number of copies of states are needed to be transformed, conversion is possible as long as the entropy of entanglement is conserved.[1]

Away from the asymptotic limit, where a single copy of a given state is to be transformed into another given state, such a simple conversion criterion cannot be found and investigations have unearthed a deep connection of the problem to the mathematical theory of majorization.[2] For setting up the necessary notation, the following definitions are introduced first. For two sequences with  $n$  elements  $x$  and  $y$ , we say that  $x$  is *super-majorized* by  $y$  (written  $x \prec^w y$ ), if  $F_m(x) \geq F_m(y)$  for all  $m = 1, 2, \dots, n$ . Here,  $F_m(x)$  denotes the sum of the smallest  $m$  elements of  $x$ , i.e.,  $F_m(x) = x_1^\uparrow + x_2^\uparrow + \dots + x_m^\uparrow$ , where  $x^\uparrow$  is the sequence  $x$  with all elements arranged in non-decreasing order ( $x_1^\uparrow \leq x_2^\uparrow \leq \dots \leq x_n^\uparrow$ ). If, in addition to these, the two sequences have the same sum ( $F_n(x) = F_n(y)$ ) then we say that  $x$  is *majorized* by  $y$  (written  $x \prec y$ ).

Given two entangled states in Schmidt form,  $|\psi\rangle = \sum_{i=1}^n \sqrt{x_i} |i_A\rangle \otimes |i_B\rangle$  and  $|\phi\rangle = \sum_{i=1}^n \sqrt{y_i} |i'_A\rangle \otimes |i'_B\rangle$ , where  $x$  and  $y$  are the respective Schmidt coefficients ( $\sum x_i = \sum y_i = 1$ ), the problem is essentially to determine the probability of converting the state  $|\psi\rangle$  into  $|\phi\rangle$  by LOCC. As two entangled states with the same Schmidt coefficients are equivalent under local unitaries, that probability depends only on the Schmidt coefficients and not on the particular local orthonormal bases in which they are expressed. For that reason, the conversion probability of  $|\psi\rangle$  into  $|\phi\rangle$  will be simply denoted by  $P(x \rightarrow y)$ .

The most important step in the solution of this problem is taken by Nielsen who has shown that  $|\psi\rangle$  can be converted into  $|\phi\rangle$  with certainty, i.e.,  $P(x \rightarrow y) = 1$ , if and only if  $x \prec y$ . [3] Subsequently, Vidal has obtained the expression  $P(x \rightarrow y) = \min_{1 \leq m \leq n} F_m(x)/F_m(y)$  for the conversion probability between two arbitrary states. [4]

Note that the conversion probability is equal to the largest value of  $\lambda$  such that  $x$  is super-majorized by  $\lambda y$ , i.e.,

$$P(x \rightarrow y) = \max \{ \lambda : \lambda \geq 0, \quad x \prec^w \lambda y \} \quad , \quad (1)$$

where  $\lambda y$  denotes the sequence obtained by multiplying each element of  $y$  with  $\lambda$ .

An interesting development came with the demonstration of Jonathan and Plenio that entangled pairs can be used just like catalysts to improve conversion probabilities. [5] To be explicit, if  $|\chi\rangle = \sum_{\ell=1}^N \sqrt{c_\ell} |\ell_A\rangle \otimes |\ell_B\rangle$  is another entangled state shared by the same parties, then for some cases  $|\psi\rangle \otimes |\chi\rangle$  can be converted into  $|\phi\rangle \otimes |\chi\rangle$  with a probability more than that of  $|\psi\rangle$  to  $|\phi\rangle$  conversion. In terms of the Schmidt coefficients we have  $P(x \otimes c \rightarrow y \otimes c) \geq P(x \rightarrow y)$ , where strict inequality is obtained for some cases. In such a transformation, the entangled state  $|\chi\rangle$  is not consumed, although it takes part in the transformation much like a catalyst in chemical reactions.

Subsequently, a lot of research has been directed to understanding the catalytic transformations. [6, 7, 8] A major problem to be solved is to determine the catalytic conversion probability, i.e.,  $P_{\text{cat}}(x \rightarrow y) = \sup_c P(x \otimes c \rightarrow y \otimes c)$ , where the supremum is taken over all finite sequences  $c$  of positive numbers. This quantity is actually the least upper bound on catalytic conversion probabilities as it may not be possible to attain the probability value  $P_{\text{cat}}(x \rightarrow y)$  by a reasonable catalyst  $c$ . However, for any probability smaller than the bound, catalysis is possible.

Nielsen has suggested the term  $x \prec_T y$  ( $x$  is *trumped* by  $y$ ) whenever there is a  $c$  such that  $x \otimes c \prec y \otimes c$ . [2] The notation will be extended and we will say that  $x$  is *super-trumped* by  $y$  (written  $x \prec_T^w y$ ) if there is a  $c$  such that  $x \otimes c \prec^w y \otimes c$ . The catalytic conversion probability can be expressed with this notation as

$$P_{\text{cat}}(x \rightarrow y) = \sup \{ \lambda : \lambda \geq 0, \quad x \prec_T^w \lambda y \} \quad . \quad (2)$$

The purpose of this letter is to provide a computable expression for that probability, mainly by finding all of the necessary and sufficient conditions for  $x \prec_T^w y$  relation for the case  $\sum x_i > \sum y_i$ . As the case  $\sum x_i = \sum y_i$  is not covered, the results in this letter will not enable us to analyze the trumping relation.

First, let us define  $\nu$ th power mean of an  $n$ -element sequence  $x$  as

$$A_\nu(x) = \left( \frac{1}{n} \sum_{i=1}^n x_i^\nu \right)^{\frac{1}{\nu}}. \quad (3)$$

For all finite  $\nu$ , this is a continuous function which has a limit  $A_{-\infty}(x) = x_1^\dagger$ . For the particular value  $\nu = 0$ , it gives the geometric mean  $A_0(x) = (\prod x_i)^{1/n}$ . Note that, if any element of the sequence  $x$  is zero, then  $A_\nu(x) = 0$  for all  $\nu \leq 0$ . We would like to prove the following theorem.

**Theorem:** If  $x$  and  $y$  are  $n$ -element sequences of non-negative numbers such that  $x$  has only positive elements and  $\sum x_i > \sum y_i$ , then  $x \prec_T^w y$  if and only if

$$A_\nu(x) > A_\nu(y) \quad , \quad \forall \nu \in (-\infty, 1) \quad . \quad (4)$$

Note that the inequalities are strict and the end point  $\nu = -\infty$  is not included ( $\nu = 1$  is also strict by assumption).

Even though the theorem deals only with the special case  $\sum x_i > \sum y_i$ , it is nevertheless possible to express the catalytic transformation probability as

$$P_{\text{cat}}(x \rightarrow y) = \min_{\nu \in [-\infty, 1]} \frac{A_\nu(x)}{A_\nu(y)} \quad . \quad (5)$$

where min is used by the inclusion of the end points. Although the minimization is over a continuous variable, it is possible to compute  $P_{\text{cat}}(x \rightarrow y)$  to any desired accuracy. Moreover, the theorem tells us that if there is a  $\nu$  in the interval  $(-\infty, 1)$  that attains the minimum of (5), then  $P_{\text{cat}}(x \rightarrow y)$  can not be achieved by any catalyst  $c$  (e.g., when  $P_{\text{cat}}(x \rightarrow y) < \min(1, x_1^\dagger/y_1^\dagger)$ ). On the other hand, if this is not the case and  $P_{\text{cat}}(x \rightarrow y) < 1$ , then that value can be achieved by some catalyst  $c$ .

The following facts, which are not too difficult to prove, will be frequently used. (1) For any sequence  $x$ , we define the *characteristic function*  $H_x(t) = \sum_{i=1}^n (t - x_i)^+$  where  $(\alpha)^+ = \max(\alpha, 0)$  denotes the positive-part function. Super-majorization relation  $x \prec^w y$  between two non-negative sequences can be equivalently stated[2] as  $H_x(t) \leq H_y(t)$  for all  $t \geq 0$ . (2) Moreover, if  $x^\dagger \neq y^\dagger$ , then  $H_y(t) - H_x(t)$  is strictly positive on some interval. (3) For the cross-product of two sequences we have  $H_{x \otimes c} = \sum_\ell c_\ell H_x(t/c_\ell)$ . (4) If all elements of  $\bar{x}$  is greater than the corresponding elements of  $x$ , i.e.,  $\bar{x}_i \geq x_i$ , then  $\bar{x} \prec^w x$ . (5) If  $x \prec^w y$  then  $x \prec_T^w y$ . (6) Finally,  $\prec^w$  and  $\prec_T^w$  are partial orders on sequences with  $n$ -elements.

*Proof of necessity:* It will be shown that if  $x \prec_T^w y$ ,  $x$  has no zero elements and  $x^\dagger \neq y^\dagger$ , then the inequalities (4) are satisfied (it is not necessary to assume  $\sum x_i > \sum y_i$ ). There is a sequence  $c$  having positive elements such that  $\Delta(t) = H_{y \otimes c}(t) - H_{x \otimes c}(t)$  is non-negative. For  $t > c_{\max} \max(x_n^\dagger, y_n^\dagger)$ , the function  $\Delta(t)$  has the constant value  $(\sum x_i - \sum y_i) \sum c_\ell$ . For that reason, the integral

$$I_\nu = \int_0^\infty \Delta(t) t^{\nu-2} dt \quad (6)$$

is convergent at  $t = \infty$  for all values of  $\nu < 1$ . Moreover, (i) if  $y$  has no zero elements, then  $\Delta(t) = 0$  for a sufficiently small  $t$  and the integral is convergent at  $t = 0$ . (ii) If  $y$  has zero entries, then  $\Delta(t) \propto t$  near  $t = 0$  and therefore the integral is convergent only for  $0 < \nu < 1$ ; but this is sufficient for us as (4) is satisfied for all  $\nu \leq 0$ . Finally, strict positivity of  $\Delta(t)$  in some interval implies that  $I_\nu$  is strictly positive. Since the integral is

$$I_\nu = \begin{cases} \frac{1}{\nu(1-\nu)} \left( \sum_{i=1}^n x_i^\nu - y_i^\nu \right) \sum_\ell c_\ell^\nu & \nu \neq 0 \\ \left( \ln \prod x_i / \prod y_i \right) \left( \sum_\ell c_\ell \right) & \nu = 0 \end{cases} \quad (7)$$

investigating  $\nu < 0$ ,  $\nu = 0$  and  $\nu > 0$  cases separately, it can be seen that (4) are satisfied.  $\square$

Proof of sufficiency is lengthy and needs the introduction of a separate problem. Let  $\gamma(s) = \sum_{m=0}^N \gamma_m s^m$  be a real polynomial where some of the coefficients  $\gamma_m$  might be negative. The problem is to express  $\gamma$  as a ratio of two power series with non-negative coefficients, which are required to be convergent at a desired value  $s = R$ . To be precise, we would like to find two power series  $a(s) = \sum_{m=0}^\infty a_m s^m$  and  $b(s) = \sum_{m=0}^\infty b_m s^m$  such that (i)  $a(s)\gamma(s) = b(s)$ , (ii)  $a_m \geq 0$  and  $b_m \geq 0$  for all  $m$  and finally (iii) both  $a(R)$  and  $b(R)$  are finite. We will say that  $\gamma$  belongs to the polynomial set  $\mathcal{P}_R$  when this problem has a solution. It is obvious that if  $\gamma \in \mathcal{P}_R$ , then  $\gamma(s) > 0$  for all  $s \in (0, R]$ . The following lemma shows that this property is also sufficient.

**Lemma:** For a polynomial  $\gamma(s)$ , if  $\gamma(s) > 0$  for all  $s$  in the range  $0 < s \leq R$  then  $\gamma \in \mathcal{P}_R$ .

*Proof:* First, note that the product of two elements of  $\mathcal{P}_R$  is in the same set. For if  $\gamma_1, \gamma_2 \in \mathcal{P}_R$  and  $a_i$  and  $b_i$  are the respective series satisfying positive coefficient and convergence properties such that  $a_i(s)\gamma_i(s) = b_i(s)$  for  $i = 1, 2$ , then we have  $a_1(s)a_2(s)\gamma_1(s)\gamma_2(s) = b_1(s)b_2(s)$ . Since  $a_1a_2$  and  $b_1b_2$  are convergent at  $R$  and have non-negative series coefficients we have  $\gamma_1\gamma_2 \in \mathcal{P}_R$ . For that reason, the assertion will first be proven for irreducible factors of  $\gamma$ .

(1) For  $\gamma(s) = 1 - \xi s$ , it will be shown that if  $\xi R < 1$  then  $\gamma \in \mathcal{P}_R$ . For the case,  $\xi \leq 0$ , there is nothing to be shown as  $\gamma$  has already non-negative coefficients. For the case,  $0 < \xi R < 1$ , we have  $a(s) = (1 - \xi s)^{-1} = \sum_{m=0}^\infty \xi^m s^m$  and  $b(s) = 1$ , which satisfy the requirements, so that we have  $\gamma \in \mathcal{P}_R$ .

(2) For  $\gamma(s) = 1 - 2\xi s + \lambda s^2$ , it will be shown that if  $\lambda > \xi^2$  then  $\gamma \in \mathcal{P}_R$ . Obviously, for  $\xi \leq 0$  there is nothing to be proven, so consider  $\xi > 0$  for the following. Let  $N$  be an integer sufficiently large so that

$$\frac{1}{4} \left( \frac{(2N)!}{N!^2} \right)^{\frac{1}{N}} \geq \frac{\xi^2}{\lambda} \quad . \quad (8)$$

We can always find such an  $N$  as the left-hand side has limit 1 as  $N \rightarrow \infty$  and the right-hand side is strictly less

than 1. In that case we choose

$$a(s) = \sum_{k=0}^{2N-1} (1 + \lambda s^2)^k (2\xi s)^{2N-1-k} \quad , \quad (9)$$

$$b(s) = (1 + \lambda s^2)^{2N} - (2\xi s)^{2N} \quad . \quad (10)$$

Note that all coefficients of  $a$  are already non-negative. That is true for  $b$  as well, since the coefficient of  $s^{2N}$  is  $\lambda^N (2N)!/N!^2 - (2\xi)^{2N}$  which is also non-negative by the special choice of  $N$ . Therefore,  $\gamma \in \mathcal{P}_R$ .

(3) The lemma can now be proven for a general polynomial. Express  $\gamma(s)$  as a product of its irreducible factors as

$$\gamma(s) = A s^r \prod_i (1 - \xi_i s) \prod_i (1 - 2\xi'_i s + (\xi_i'^2 + \eta_i'^2) s^2) \quad , \quad (11)$$

where  $r(\geq 0)$  is the multiplicity of a possible root at 0,  $1/\xi_i$  are the real roots,  $(\xi_i' \pm i\eta_i')^{-1}$  are the complex roots of  $\gamma$  and  $A > 0$ . Since  $\gamma$  is non-zero on the interval  $(0, R]$ , each real root satisfies  $\xi_i R < 1$ . As each factor is in  $\mathcal{P}_R$ , we have  $\gamma \in \mathcal{P}_R$ .  $\square$

Note that if  $R > 1$  and  $\gamma \in \mathcal{P}_R$ , then the infinite series  $a(s)$  can be chosen such that the value  $a(1)$  and all series coefficients  $a_m$  are rational numbers. The reason is that  $a(s)$  and  $b(s)$  can both be multiplied by a third series which satisfies the necessary non-negativity and convergence properties. By choosing the coefficients of the third series, all of these numbers can be made rational simultaneously as the reader can easily check. After this brief diversion, we can continue with the rest of the proof of the theorem.

*Proof of sufficiency:* If two  $n$ -element sequences  $x$  and  $y$  (such that  $x^\uparrow \neq y^\uparrow$ ) share some common elements, then the corresponding elements can be removed from each, which gives shorter sequences  $\bar{x}$  and  $\bar{y}$  (which have no common elements, i.e.,  $\bar{x}_i \neq \bar{y}_j$ ). It is easy to verify that (i)  $x \prec^w y$  iff  $\bar{x} \prec^w \bar{y}$ , (ii)  $x \prec_T^w y$  iff  $\bar{x} \prec_T^w \bar{y}$ , and (iii)  $A_\nu(x) > A_\nu(y)$  iff  $A_\nu(\bar{x}) > A_\nu(\bar{y})$ . For this reason, it is only necessary to give the proof for sequences which have no common elements. This will be assumed below. It will also be assumed that  $x$  and  $y$  are arranged in non-decreasing order ( $x = x^\uparrow$  and  $y = y^\uparrow$ ). The complete proof of the sufficiency of the inequalities (4) will be completed in three steps, each one being in the form of a separate theorem dealing with a special case.

**Case A.**  $y$  has strictly positive elements such that  $y_i = K\omega^{\alpha_i}$  and  $x_i = K\omega^{\beta_i}$  for some integers  $\alpha_i$  and  $\beta_i$  and for some numbers  $K > 0$  and  $\omega > 1$ .

*Proof:* Redefine  $K$  such that  $\alpha_1 = 0$  (as a result,  $\alpha_i \geq 0$  for all  $i$ ) and then set  $K = 1$  by dividing each sequence by a common number. Note that  $\nu \rightarrow -\infty$  limit of (4) gives  $x_1 \geq y_1$ . As  $x$  and  $y$  have no common elements, we have  $\beta_i > 0$  for all  $i$ . Let the polynomial  $\Gamma(s)$  be defined

as

$$\Gamma(s) = \sum_{i=1}^n (s^{\alpha_i} - s^{\beta_i}) = \sum_k \Gamma_k s^k \quad , \quad (12)$$

and let  $\gamma(s) = \Gamma(s)/(1-s)$ . Since  $\Gamma(1) = 0$ ,  $\gamma(s)$  is also a polynomial. We will first show that  $\gamma \in \mathcal{P}_\omega$ . The inequality (4) at  $\nu = 0$  implies that  $\gamma(1) = \sum_{i=1}^n (\beta_i - \alpha_i)$  is strictly positive. Next, let  $s = \omega^\nu$  where  $\nu$  is any value in  $(-\infty, 1]$  excluding  $\nu = 0$ . In that case, we have

$$\gamma(s) = \frac{1}{1-\omega^\nu} \sum_{i=1}^n (y_i^\nu - x_i^\nu) \quad . \quad (13)$$

Investigating the cases  $\nu < 0$  and  $\nu > 0$  separately, one finds that  $\gamma(s) > 0$ . As a result, we have  $\gamma \in \mathcal{P}_\omega$ .

By the lemma, there exists two (possibly infinite) series  $a(s)$  and  $b(s)$  which are convergent at  $s = \omega$  and have non-negative series coefficients. Moreover,  $a(s)$  will be chosen in such a way that all of its coefficients and  $a(1)$  are rational numbers. As  $\gamma(0) > 0$ ,  $a_0$  and  $b_0$  can be made non-zero. The relationship  $a(s)\Gamma(s) = (1-s)b(s)$  implies that  $\sum_{k=0}^m a_k \Gamma_{m-k} = b_m - b_{m-1}$ , where we define  $b_{-1} = 0$  for simplicity.

Let  $\bar{h}(t) = \sum_{m=0}^\infty a_m (t - \omega^m)^+$ , a function which is a sum of a finite number of terms for any fixed  $t$ . Let

$$\begin{aligned} \bar{\delta}(t) &= \sum_{i=1}^n y_i \bar{h}\left(\frac{t}{y_i}\right) - x_i \bar{h}\left(\frac{t}{x_i}\right) = \sum_k \Gamma_k \omega^k \bar{h}(t\omega^{-k}) \quad , \\ &= \sum_{m=0}^\infty (b_m - b_{m-1})(t - \omega^m)^+ \quad . \end{aligned} \quad (14)$$

It can be shown that  $\bar{\delta}(t) \geq 0$  for all  $t \geq 0$ , but better lower bounds can be placed as follows: (i) For  $t \leq \omega$ , we have  $\bar{\delta}(t) = b_0(t-1)^+ \geq 0$ . (ii) For  $t \geq \omega$ , there is an integer  $N \geq 1$  such that  $\omega^N \leq t \leq \omega^{N+1}$  and we have

$$\bar{\delta}(t) = b_N(t - \omega^N) + (\omega - 1) \sum_{m=0}^{N-1} b_m \omega^m \geq (\omega - 1)b_0 \quad (15)$$

i.e., a strictly positive lower bound.

Let  $\epsilon = (\omega - 1)b_0/(\sum_k |\Gamma_k| \omega^k)$ . Since  $a(\omega) < \infty$ , we can find an integer  $M(\geq 1)$  such that  $\sum_{m=M}^\infty a_m \omega^m < \epsilon/2$ . Define  $A = \sum_{m=M}^\infty a_m$ . This is a rational number and satisfies the inequality  $A\omega^M < \epsilon/2$ . Consider the function

$$h(t) = \sum_{m=0}^{M-1} a_m (t - \omega^m)^+ + A(t - \omega^M)^+ \quad . \quad (16)$$

The following bounds can be placed on  $|\bar{h}(t) - h(t)|$ : (i) If  $t \leq \omega^M$  we have  $h(t) = \bar{h}(t)$ . (ii) If  $t \geq \omega^M$ , there is an  $N \geq M$  such that  $\omega^N \leq t \leq \omega^{N+1}$  and

$$\begin{aligned} |\bar{h}(t) - h(t)| &= \left| A\omega^M - \sum_{m=M}^N a_m \omega^m - \sum_{m=N+1}^\infty a_m t \right| \\ &\leq A\omega^M + \sum_{m=M}^\infty a_m \omega^m < \epsilon \quad . \end{aligned} \quad (17)$$

As a result, the following function

$$\begin{aligned} \delta(t) &= \sum_{i=1}^n y_i h\left(\frac{t}{y_i}\right) - x_i h\left(\frac{t}{x_i}\right) = \sum_k \Gamma_k \omega^k h(t\omega^{-k}) \quad , \\ &= \bar{\delta}(t) + \sum_k \Gamma_k \omega^k (h(t\omega^{-k}) - \bar{h}(t\omega^{-k})) \end{aligned} \quad (18)$$

is non-negative everywhere since (i) for  $t \leq \omega$  we have  $\delta(t) = \bar{\delta}(t) \geq 0$  and (ii) for  $t \geq \omega$  we have  $\delta(t) > \bar{\delta}(t) - \sum_k |\Gamma_k| \omega^k \epsilon = \bar{\delta}(t) - (\omega - 1)b_0 \geq 0$ .

Let  $\mathcal{N}$  be a sufficiently large integer so that all of  $\mathcal{N}a_0, \mathcal{N}a_1, \dots, \mathcal{N}a_{M-1}, \mathcal{N}A$  are integers. Schmidt coefficients of the catalyst sequence  $c$  will be chosen as  $\omega^m$ , repeated  $\mathcal{N}a_m$  times (for  $0 \leq m \leq M-1$ ), and as  $\omega^M$ , repeated  $\mathcal{N}A$  times. Then  $H_c(t) = \mathcal{N}h(t)$  is the characteristic function of  $c$  and the non-negativity of  $\delta(t)$  is equivalent to  $x \otimes c \prec_T^w y \otimes c$ . This proves our assertion that  $x \prec_T^w y$ .  $\square$

**Case B.**  $y$  has strictly positive elements.

*Proof:* As  $x$  and  $y$  have no common elements, the inequalities (4) imply that  $x_1^\uparrow > y_1^\uparrow$ . Let,  $\theta = \min_{\nu \in [-\infty, 1]} A_\nu(x)/A_\nu(y)$ . Since the end points are included, the minimum exists and therefore  $\theta > 1$ . Let  $\omega = \theta^{1/3}$  and define two  $n$ -element sequences  $\bar{x}$  and  $\bar{y}$  as  $\bar{y}_i = \omega^{\alpha_i}$  and  $\bar{x}_i = \omega^{\beta_i}$  where

$$\alpha_i = \left\lfloor \frac{\ln y_i}{\ln \omega} \right\rfloor, \quad \beta_i = \left\lceil \frac{\ln x_i}{\ln \omega} \right\rceil, \quad (19)$$

$\lfloor t \rfloor$  is the largest integer smaller than  $t$  and  $\lceil t \rceil$  is the smallest integer greater than  $t$ . Using  $\lfloor t \rfloor - 1 < t \leq \lceil t \rceil$  and  $\lfloor t \rfloor \leq t < \lceil t \rceil + 1$ , we get

$$\frac{\bar{y}_i}{\omega} < y_i \leq \bar{y}_i, \quad \bar{x}_i \leq x_i < \omega \bar{x}_i. \quad (20)$$

Then for any  $\nu \in [-\infty, 1]$  we have

$$A_\nu(\bar{x}) > \frac{1}{\omega} A_\nu(x) \geq \frac{\theta}{\omega} A_\nu(y) > \frac{\theta}{\omega^2} A_\nu(\bar{y}). \quad (21)$$

As a result,  $A_\nu(\bar{x}) > A_\nu(\bar{y})$  for all  $\nu \in [-\infty, 1]$ ;  $\bar{x}$  and  $\bar{y}$  fulfills the conditions of case A, and therefore  $\bar{x} \prec_T^w \bar{y}$ . Finally, the inequalities (20) imply  $x \prec_T^w \bar{x}$  and  $\bar{y} \prec_T^w y$ . All of these prove our assertion that  $x \prec_T^w y$ .  $\square$

**Case C.**  $y$  has zero elements.

The proof will be carried out by replacing all zero elements of  $y$  with a small value  $\epsilon$  in such a way that this case is reduced to case B. Suppose that  $y$  has exactly  $m$  entries equal to 0 ( $0 < m < n$ ). Note that the inequalities (4) are automatically satisfied for  $\nu \leq 0$ . Using the premise that (4) are satisfied for  $\nu \in (0, 1]$ , we can deduce that the function

$$J_\nu = \left( \frac{\sum_{i=1}^n x_i^\nu - \sum_{i=m+1}^n y_i^\nu}{m} \right)^{\frac{1}{\nu}}, \quad (22)$$

is strictly positive for all  $\nu \in (0, 1]$ . Moreover, it has a positive limit  $J_0 = (\prod x / \prod_{i=m+1}^n y_i)^{1/m}$  at the end

point  $\nu = 0$ . As a result,  $J_{\min} = \min_{\nu \in [0, 1]} J_\nu$  exists and is non-zero as the minimum is taken over a compact interval. Let  $\epsilon$  be a positive number such that

$$\epsilon < \min \left( J_{\min}, y_n \left( \frac{x_1}{y_n} \right)^{\frac{n}{m}} \right), \quad (23)$$

and define a new sequence  $\bar{y}$  as  $\bar{y}_1 = \dots = \bar{y}_m = \epsilon$  and  $\bar{y}_i = y_i$  for all  $i > m$ . It is obvious that  $\bar{y} \prec_T^w y$ . Showing that  $x \prec_T^w \bar{y}$  will complete the proof. For this purpose, we look at the power means. (i) For  $\nu \in (0, 1]$ , it is trivial to check that  $J_\nu > \epsilon$  is equivalent to  $A_\nu(x) > A_\nu(\bar{y})$ . (ii) For  $\nu = 0$  we have

$$\frac{A_0(x)}{A_0(\bar{y})} = \frac{(\prod_{i=1}^n x_i)^{\frac{1}{n}}}{(\epsilon^m \prod_{i=m+1}^n y_i)^{\frac{1}{n}}} \geq \frac{x_1}{y_n} \left( \frac{y_n}{\epsilon} \right)^{\frac{m}{n}} > 1. \quad (24)$$

(iii) For  $\nu < 0$ , we use Bernoulli's inequality, which states that  $\alpha^r - 1 \geq r(\alpha - 1)$  for all  $r \geq 1$  and  $\alpha > 0$ , as follows

$$m(\epsilon^\nu - y_n^\nu) > m y_n^\nu \left( \left( \frac{x_1}{y_n} \right)^{\frac{\nu}{m}} - 1 \right) \quad (25)$$

$$\geq m y_n^\nu \frac{n}{m} \left( \left( \frac{x_1}{y_n} \right)^\nu - 1 \right) \quad (26)$$

$$= n(x_1^\nu - y_n^\nu), \quad (27)$$

which implies that

$$\sum_{i=1}^n \bar{y}_i^\nu = m\epsilon^\nu + \sum_{i=m+1}^n y_i^\nu \geq m\epsilon^\nu + (n-m)y_n^\nu \quad (28)$$

$$> n x_1^\nu \geq \sum_{i=1}^n x_i^\nu \quad (29)$$

The result  $A_\nu(x) > A_\nu(\bar{y})$  follows from here. As power mean inequalities are satisfied for all  $\nu \in (-\infty, 1]$ , we have  $x \prec_T^w \bar{y}$  by the result in case B, which completes the proof.  $\square$

---

\* Electronic address: sturgut@metu.edu.tr

- [1] C.H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
- [2] M. A. Nielsen, *Majorization and its Applications in Quantum Information Theory*, preprint.
- [3] M. A. Nielsen, Phys. Rev. Lett. **83** 436 (1999).
- [4] G. Vidal, Phys. Rev. Lett. **83** 1046 (1999).
- [5] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83** 3566 (1999).
- [6] R. Duan, Z. Ji, Y. Feng, and M. Ying, J. Comput. Sci. & Technol. **21** 776 (2006).
- [7] S. Daftuar and M. Klimesh, Phys. Rev. A **64** 042314 (2001).
- [8] Y. Feng, R. Duan, and M. Ying, Phys. Rev. A **69** 062310 (2004); X. Sun, R. Duan, and M. Ying, IEEE Trans. Inf. Theory **51** 75 (2005); Y. Feng, R. Duan, and M. Ying, IEEE Trans. Inf. Theory **51** 1090 (2005).